A FORMULA OF POLYNOMIAL INTERPOLATION.*

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Many proofs have been given of the following theorem of Weierstrass:†

Let $f(x)$ be a real function, defined and continuous for $a \leqq x \leqq b$; and let ϵ be an arbitrary positive constant. Then there exists a polynomial $G(x)$ such that for $a \leqq x \leqq b$

$$| f(x) - G(x) | < \epsilon.$$

Of these proofs one of the simplest is due to Landau.‡ A proof will be proposed here, which, while closely related to Landau's, is perhaps more elementary, in that integrals are replaced by finite sums; and at the same time it retains something of the elegance for which Landau's proof is conspicuous above others that are more elementary still.§ The formula used here has some interest for its own sake, as a formula of interpolation, and as such may be compared, for example, with those of Borel,|| Faber,¶ and Runge.**

Without loss of generality, we may assume that $0 < a < b < 1$, since a linear transformation can be performed on x, if necessary, without affecting the essential conditions of the problem. Let the definition of $f(x)$ be extended by setting $f(x) = f(a)$ for $0 \leqq x \leqq a$, $f(x) = f(b)$ for $b \leqq x \leqq 1$. Then $f(x)$ is defined and continuous for $0 \leqq x \leqq 1$. We shall now prove the

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* This paper, as originally written, contained in addition to the polynomial formula given here a trigonometric formula of approximation. While this paper was in the hands of the editors, Kryloff published in Bulletin des sciences mathématiques, second series, vol. 41 (1917), pp, 309–320, a paper in which he establishes the validity of the same formula. So the reader is referred to Kryloff's article for the trigonometric formula.

† Weierstrass, "Über die analytische Darstellbarkeit sogennanter willkürlicher Functionen einer reellen Veränderlichen," Sitzungsberichte der Kgl. Preuss. Akademie der Wissenschaften, 1885, pp. 633–639, 789–805. "Über die analytische Darstellbarkeit sogennanter willkürlicher Functionen reeller Argumente," Werke, vol. 3 (1903), pp. 1–37.

‡ Landau, "Über die Approximation einer stetigen Funktion durch eine ganze rationale Funktion," Rendiconti del Circolo Matematico di Palermo, vol. 25 (1908), pp. 337–345.

§ Cf., e. g., Lebesgue, "Sur l'approximation des fonctions," Bulletin des sciences mathématiques, series 2, vol. 22 (1898), pp. 278–287.

|| Borel, Leçons sur les fonctions de variables réelles et les développements en séries de polynomes. Paris, 1905, pp. 79–82.

¶ Faber, "Über stets konvergente Interpolationsformeln," Jahresbericht der deutschen Mathematiker-Vereinigung, vol. 19 (1910), pp. 142–146.

** Runge, Theorie und Praxis der Reihen, pp. 134–142.

THEOREM. *When n increases indefinitely, the polynomial in x (of degree $2n$)*

$$(1) \qquad \sigma_n(x) = \frac{\sum_{i=0}^{n} f(t_i)[1 - (t_i - x)^2]^n}{2\sum_{i=0}^{n} (1 - t_i^2)^n},$$

where $t_i = i/n$, approaches the limit $f(x)$ uniformly for $a \leqq x \leqq b$.

First consider the case when $f(x)$ is identically equal to *unity*, and write

$$(2) \qquad \tau_n(x) = \frac{\sum_{i=0}^{n} [1 - (t_i - x)^2]^n}{2\sum_{i=0}^{n} (1 - t_i^2)^n},$$

In this expression, the numerator is less than the denominator. For each of the two largest terms of the numerator is less than or equal to unity, each of the two next in order of magnitude is less than or equal to $(1 - t_1^2)^n$, and so on, while in the denominator there will be a surplus of terms which will not be called into comparison in this way at all. Therefore

$$(3) \qquad \tau_n(x) < 1.$$

On the other hand, each of the two largest terms of the numerator of (2) is at least equal to $(1 - t_1^2)^n$, each of the next two in order of magnitude is at least equal to $(1 - t_2^2)^n$, and so on for at least k pairs of terms, if x is in the interval (a, b) and k, depending on n, is the greatest integer such that k/n is less than γ, γ being a positive quantity such that $0 < a - \gamma < b + \gamma < 1$. Then

$$\tau_n(x) > \frac{2\sum_{i=1}^{k} (1 - t_i^2)^n}{2\sum_{i=0}^{n} (1 - t_i^2)^n} = 1 - \frac{1 + \sum_{i=k+1}^{n} (1 - t_i^2)^n}{\sum_{i=0}^{n} (1 - t_i^2)^n}.$$

For $i > k$, $1 - t_i^2 < 1 - \gamma^2$. Therefore

$$(4) \qquad \tau_n(x) > 1 - \frac{1 + (n - k)(1 - \gamma^2)^n}{\sum_{i=0}^{n} (1 - t_i^2)^n} > 1 - \frac{1 + n(1 - \gamma^2)^n}{\sum_{i=0}^{n} (1 - t_i^2)^n}.$$

Since evidently $0 < 1 - \gamma^2 < 1$, $n(1 - \gamma^2)^n$ approaches zero as a limit as n increases indefinitely. Now

$$\lim_{n \to \infty} \sum_{i=0}^{n} (1 - t_i^2)^n = \lim_{n \to \infty} \sum_{i=0}^{n} \left(1 - \frac{i^2}{n^2}\right)^n = \infty,$$

as the following shows.

Out of $\sum_{i=0}^{n}(1 - i^2/n^2)^n$ consider only the first $[\sqrt{n}]$ terms.* Each of these terms will be decreased if n is put in place of i^2, so that

$$\sum_{i=0}^{n}(1 - i^2/n^2)^n > [\sqrt{n}](1 - 1/n)^n > K[\sqrt{n}],$$

where K is independent of n, since $(1 - 1/n)^n$ approaches a limit, not zero, as n increases indefinitely. Hence $\sum_{i=0}^{n}(1 - t_i^2)^n$ increases indefinitely with n.

On combination of this result with (4) and (3), it is seen at once that to every $\epsilon > 0$ there corresponds an N independent of x such that

$$1 - \epsilon < \tau_n(x) < 1, \qquad n \geqq N.$$

Therefore when $a \leqq x \leqq b$, $\tau_n(x)$ approaches unity uniformly as n increases indefinitely.

Now let $f(x)$ be the arbitrary function for which the theorem is stated. Let

$$\varphi_n(x) = f(x)\tau_n(x) = \frac{\sum_{i=0}^{n} f(x)[1 - (t_i - x)^2]^n}{2\sum_{i=0}^{n}(1 - t_i^2)^n}. \tag{5}$$

It is an immediate consequence of the preceding that $\varphi_n(x)$ converges uniformly to the limit $f(x)$. Thus, given a positive ϵ, it is possible to find an n_1, independent of x, such that

$$|\varphi_n(x) - f(x)| < \epsilon/3, \qquad n \geqq n_1. \tag{6}$$

On the other hand, by subtracting (5) from (1) and taking absolute values, we see that

$$|\sigma_n(x) - \varphi_n(x)| \leqq \frac{\sum_{i=0}^{n}|f(t_i) - f(x)|[1 - (t_i - x)^2]^n}{2\sum_{i=0}^{n}(1 - t_i^2)^n}. \tag{7}$$

Since $f(x)$ is continuous on the closed interval $0 \leqq x \leqq 1$, we can find a $\delta < 1$ such that for any two values x', x'', in the interval considered,

$$|f(x'') - f(x')| < \epsilon/3, \text{ when } |x'' - x'| < \delta.$$

It may be assumed further that $\delta < \gamma$. The expression on the right of (7) can be broken up into two parts, as follows. Let the terms in the numerator for which $-\delta \leqq t_i - x \leqq \delta$ be added to form the numerator

* $[\sqrt{n}]$ stands for the greatest integer in $\sqrt{n}$.

of a new fraction Σ_n' with same denominator, and let the remaining ones be similarly used to form a fraction Σ_n''. It is assumed now that x is in the interval $a \leqq x \leqq b$. The sum Σ_n' will be increased if $\epsilon/3$ is put in place of $|f(t_i) - f(x)|$. The resulting expression with the factor $\epsilon/3$ removed contains only a part of the terms of $\tau_n(x)$, all of which are positive. Thus

$$\Sigma_n' < \epsilon/3,$$

in consequence of (3). In Σ_n'', $[1 - (t_i - x)^2]^n < (1 - \delta^2)^n < 1$, and replacing each difference $|f(t_i) - f(x)|$ by $2M$, where M is the maximum of the absolute value of $f(x)$, we obtain

$$\Sigma_n'' < \frac{2M(n+1)(1-\delta^2)^n}{2\sum_{i=0}^{n}(1-t_i^2)^n}.$$

The numerator approaches zero as n increases indefinitely, and the denominator becomes infinite. Then for a sufficiently large n_2, independent of x,

$$\Sigma_n'' < \epsilon/3, \qquad n \geqq n_2; \qquad \Sigma_n' + \Sigma_n'' < 2\epsilon/3, \qquad n \geqq n_2,$$

and

$$|\sigma_n(x) - \varphi_n(x)| < 2\epsilon/3, \qquad n \geqq n_2.$$

On combining this with (6), and taking n_0 equal to the greater of n_1 and n_2, we finally obtain

$$|\sigma_n(x) - f(x)| < \epsilon, \qquad n \geqq n_0.$$

Therefore $\sigma_n(x)$ approaches the limit $f(x)$ uniformly as n increases indefinitely.

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